Proof of a conjecture of Okada

Greta Panova

December 5, 2008

Abstract

We prove an equation conjectured by Okada regarding hook-lengths of partitions, namely that

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 \sum_{u\in\lambda}\prod_{i=1}^{r}(h_u^2-i^2) = \frac{1}{2(r+1)^2}\binom{2r}{r}\binom{2r+2}{r+1}\prod_{j=0}^{r}(n-j),$$

where f_λ is the number of standard Young tableaux of shape λ and h_u is the hook length of the square u of λ. We also obtain other similar formulas.

1 Introduction

If F is any symmetric function then define

$$\Phi_n(F) = \frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 F(h_u^2 : u\in\lambda), \tag{1}$$

where the sum runs over all partitions λ of n, f_λ is the number of standard Young tableaux of shape λ and h_u denotes the hook length of the square u in that partition. In [2] Stanley proves that $\Phi_n(F)$ is a polynomial in n. Following this theorem Soichi Okada conjectured an explicit formula (see [2]).

Theorem 1. (Okada's conjecture) For every integer $n \geq 1$ and every nonnegative integer r we have that

$$\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 \sum_{u\in\lambda}\prod_{i=1}^{r}(h_u^2-i^2) = \frac{1}{2(r+1)^2}\binom{2r}{r}\binom{2r+2}{r+1}\prod_{j=0}^{r}(n-j). \tag{2}$$

The current note is devoted to proving this equation and similar results. In doing so we also prove a conjecture by G. Han from [6] and generalize his "marked hook formula" from [3].

2 Proof of Okada's conjecture

Let $P_r(n) = \frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 \sum_{u\in\lambda}\prod_{i=1}^r (h_u^2 - i^2)$. Since $F_r(x_1,\ldots,x_n) = \sum_{i=1}^n \prod_{j=1}^r (x_i - j^2)$ is clearly symmetric in the variables $x_1,\ldots,x_n$, we see that $P_r(n) = \frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 F_r(h_u^2 : u\in\lambda)$ and so by [2] it is a polynomial in n. In order to prove (2) then it suffices to show that the degree of the polynomial is less than or equal to $r+1$, and exhibit (2) for $r+2$ values of n.

Lemma 1. The values $1,\ldots,r$ are roots of $P_r(n)$. Moreover

$$P_r(r+1) = \frac{1}{2(r+1)^2}\binom{2r}{r}\binom{2r+2}{r+1}(r+1)!$$

and

$$P_r(r+2) = \frac{1}{2(r+1)^2}\binom{2r}{r}\binom{2r+2}{r+1}(r+2)!$$

.

Proof. If $1\le n\le r$ we have that for every $\lambda\vdash n$ and every $u\in\lambda$ that $1\le h_u\le|\lambda| = n\le r$, and so $\prod_{i=1}^r (h_u^2 - i^2) = (h_u^2-1^2)\cdots(h_u^2-h_u^2)\cdots(h_u^2-r^2) = 0$. Hence

$$P_r(n) = \frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 \sum_{u\in\lambda}\prod_{i=1}^r (h_u^2-i^2) = \frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 \sum_{u\in\lambda} 0 = 0$$

for $n = 1,\ldots,r$.

Now let $n = r+1$. Let $\lambda\vdash r+1$ and consider the largest hook length in λ, that is, $h_{(1,1)} = \lambda_1 + \ell(\lambda) - 1$, where $\ell(\lambda)$ denotes the number of parts of λ. If $h_{(1,1)}\le r$, then for every $u\in\lambda$ we would have $h_u\le h_{(1,1)}\le r$ and so by the argument in the previous paragraph we will have $\sum_{u\in\lambda}\prod_{i=1}^r (h_u^2 - i^2) = 0$. Hence we need to consider only λ such that $h_{(1,1)}\ge r+1$. If $\lambda\vdash r+1$ and $h_{(1,1)}\ge r+1$ we must necessarily have $h_{(1,1)} = r+1$ with all squares of λ being part of that hook, i.e., $\lambda = (a+1,\underbrace{1,\ldots,1}_{r-a})$, a hook shape with $\lambda_1 = a+1$, for some $r\ge a\ge 0$. For such λ we have by the hook-length formula, or by a simple bijection with subsets of $[2,\ldots,r+1]$ of a elements for the ones in $(1,2),\ldots,(1,a+1)$, that $f_\lambda = \frac{(r+1)!}{(r+1)a!(r-a)!} = \binom{r}{a}$. We also have that the only square u with hook length greater than r is $(1,1)$, and for it we have $\prod_{j=1}^r (h_{(1,1)}^2 - j^2) = \prod_{j=1}^r (r+1-j)\prod_{j=1}^r (r+1+j) = \frac{(2r+1)!}{r+1}$. Thus we can compute

$$\begin{aligned}
P_r(r+1) &= \frac{1}{(r+1)!}\sum_{a=0}^r f_{(a+1,1,\ldots,1)}^2 \frac{(2r+1)!}{r+1}\\
&= \frac{1}{2(r+1)^2}\binom{2r+2}{r+1}(r+1)!\sum_{a=0}^r \binom{r}{a}^2\\
&= \frac{1}{2(r+1)^2}\binom{2r}{r}\binom{2r+2}{r+1}(r+1)!,
\end{aligned}$$

which also agrees with (2).

Computing $P_r(r+2)$ is slightly more complicated, because there are two kinds of shapes λ which contain squares of hook length at least $r+1$. Since the largest hook length is $h_{(1,1)}$ we need to consider the cases $h_{(1,1)} = r+2$ and $h_{(1,1)} = r+1$. The first one implies that λ is a hook, i.e. $(a+1,1,\ldots,1)$, and the only hook of length at least $r+1$ is at $(1,1)$ unless $a=0$ or $a=r+1$, when there are additional hooks of length $r+1$ at $(2,1)$ and $(1,2)$ respectively. Hence the contribution to $P_r(r+2)$ will be

$$\frac{1}{(r+2)!}\frac{(2r+2)!}{r+2}\sum_{a=0}^{r+1}\binom{r+1}{a}^2 + \\ + f^2_{(1,1,1,\ldots,1)}\prod_{j=1}^{r}((r+1)^2-j^2) + f^2_{(r+2)}\prod_{j=1}^{r}((r+1)^2-j^2) \\ = \frac{(2r+2)!}{(r+2)!}\binom{2r+2}{r+1} + 2\frac{(2r+1)!}{(r+1)(r+2)!}. \quad (3)$$

Next, if $h_{(1,1)} = r+1$ then λ must necessarily be of shape $(a+2,2,1,\ldots,1)$ for some $a \in [0,\ldots,r-2]$. In this case $h_{(1,1)} = r+1$ and all other hook lengths are less than $r+1$, so contribute 0 to F_r. Hence $F_r = \prod_{j=1}^{r}((r+1)^2-j^2)$. We have by the hook-length formula and some algebraic manipulations of binomial coefficients that

$$f_{(a+2,2,1,\ldots,1)} = \frac{(r+2)!}{(r+1)(a+2)a!(r-a)(r-a-2)!} = (r+2)\binom{r}{a+1} - \binom{r+2}{a+2}.$$

Now we can compute the contribution of such partitions to the sum in $P_r(r+2)$ as

$$\frac{1}{(r+2)!}\sum_{a=0}^{r-2} f^2_{(a+2,2,1,\ldots,1)}\prod_{j=1}^{r}((r+1)^2-j^2) \\ = \frac{(2r+1)!}{(r+2)!(r+1)}\left(-\left((r+2)\binom{r}{0} - \binom{r+2}{1}\right)^2 \right. \\ \left. - \left((r+2)\binom{r}{r} - \binom{r+2}{r+1}\right)^2\right) + \\ + \frac{(2r+1)!}{(r+2)!(r+1)}\sum_{a=-1}^{r-1}\left((r+2)\binom{r}{a+1} - \binom{r+2}{a+2}\right)^2 \\ = \frac{(2r+1)!}{(r+2)!(r+1)}\left((r+2)^2\binom{2r}{r} - 2(r+2)\binom{2r+2}{r+1} + \binom{2r+4}{r+2} - 2\right). \quad (4)$$

Now we can finally obtain $P_r(r+2)$ as the sum of (3) and (4). After some algebraic manipulations we get

$$P_r(r+2) = \binom{2r}{r}\binom{2r+2}{r+1}\frac{1}{2(r+1)^2}(r+2)!,$$

as claimed.

$\square$

The following result was originally conjectured by Han in [6, Conjecture 3.1].

Lemma 2. The degree of the polynomial $R_k(n) = \frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 \sum_{u\in\lambda} h_u^{2k}$ is $k+1$.

Proof of Lemma 2 and equation (2). The idea for this proof is suggested by Richard Stanley. The point is to use the bijection given by the RSK algorithm between pairs of standard Young tableaux (P,Q) of same shape $\lambda \vdash n$ and permutations $w \in S_n$ (see for example [1]), together with some permutation statistics. We are going to show that $0 < \lim_{n\to\infty} \frac{R_k(n)}{n^{k+1}} < \infty$.

By the fact that the number of pairs (P,Q) of SYT's of the same shape $\mathrm{sh}(P) = \mathrm{sh}(Q) = \lambda \vdash n$ is f_λ^2 and then by the RSK algorithm between such pairs and permutations of n letters, we can rewrite $R_k(n)$ as

$$\begin{aligned} R_k(n) &= \frac{1}{n!}\sum_{\lambda\vdash n} \sum_{\substack{(P,Q),\\ \mathrm{sh}(P)=\mathrm{sh}(Q)=\lambda}} \sum_{u\in\lambda} h_u^{2k} \\ &= \frac{1}{n!} \sum_{\substack{(P,Q),\\ \mathrm{sh}(P)=\mathrm{sh}(Q)\vdash n}} \sum_{u\in \mathrm{sh}(P)} h_u^{2k} \\ &= \frac{1}{n!}\sum_{w\in S_n}\sum_{u\in\mathrm{sh}(w)} h_u^{2k}, \end{aligned} \tag{5}$$

where $\mathrm{sh}(w)$ denotes the shape of the SYTs obtained from w by the RSK algorithm, i.e., if $(P_w, Q_w) = RSK(w)$, then $\mathrm{sh}(w) = \mathrm{sh}(P_w) = \mathrm{sh}(Q_w)$.

We have that $h_{(1,1)} = \lambda_1 + \lambda_1' - 1$. Since for any $u \in \lambda$, $h_u \le h_{(1,1)}$, we have also that

$$h_u^{2k} \le (\lambda_1 + \lambda_1' - 1)^{2k} \le 2^{2k}\lambda_1^{2k} + 2^{2k}(\lambda_1')^{2k}.$$

By [1, Cor.7.23.11] we have that $\lambda_1 = \mathrm{is}(w)$, where $\mathrm{is}(w)$ denotes the length of the longest increasing subsequence of w. Hence $R_k(n)$ can be bounded as follows:

$$\begin{aligned} \frac{1}{n!}\sum_{w\in S_n} \mathrm{is}(w)^{2k} &\le \frac{1}{n!}\sum_{w\in S_n}\sum_{u\in\mathrm{sh}(w)} h_u^{2k} \\ &\le \frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 n (2^{2k}\lambda_1^{2k} + 2^{2k}(\lambda_1')^{2k}) \\ &= \frac{1}{n!} n 2^{2k} \sum_{\lambda\vdash n} f_\lambda \lambda_1^{2k} + \frac{1}{n!} n 2^{2k} \sum_{\lambda'\vdash n} f_{\lambda'} (\lambda_1')^{2k} \\ &= 2^{2k+1} n \frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda \lambda_1^{2k} = 2^{2k+1} n \frac{1}{n!}\sum_{w\in S_n} \mathrm{is}(w)^{2k}, \end{aligned} \tag{6}$$

where we also used the obvious fact that $f_{\lambda'} = f_\lambda$, so that the sums over λ and λ' become equal.

Now that we have bounded $R_k(n)$ by sums involving only permutations, we can apply some permutations statistics to obtain bounds for these sums. In [5] Hammersley proves that for uniformly distributed $w \in S_n$, the value $\mathrm{is}(w)/\sqrt{n}$ converges to a

constant c in probability and also in L_p norm for any p. In other words for any $p > 0$ there is a constant E_p such that

$$\lim_{n\to\infty} \sum_{w\in S_n} \frac{1}{n!} \left(\frac{\operatorname{is}(w)}{\sqrt{n}} \right)^p = E_p. \tag{7}$$

Thus the L_p norm (also called p^{th} moment) of $\frac{\operatorname{is}(w)}{\sqrt{n}}$ is bounded. In other words for any nonnegative k there is a constant M_k such that

$$\frac{1}{n^{k/2}} \sum_{w\in S_n} \frac{1}{n!} \operatorname{is}(w)^k < M_k.$$

By this fact and by the bounds in (6) we see that

$$R_k(n) \leq 2^{2k+1} n \sum_{w\in S_n} \frac{1}{n!} \operatorname{is}(w)^{2k} \leq 2^{2k+1} n M_{2k} n^k = 2^{2k+1} M_{2k} n^{k+1},$$

so that we must necessarily have that $\deg R_k(n) \leq k+1$ for every k. Since $\prod_{j=1}^k (h_u^2 - j^2) \leq h_u^{2k}$ we have that $P_k(n) \leq R_k(n)$, so $\deg P_k(n) \leq \deg R_k(n)$, and in particular $\deg P_k(n) \leq k+1$. In Lemma 1 we showed that $P_k(n)$ coincides with the polynomial $\frac{1}{2(k+1)^2}\binom{2k}{k}\binom{2k+2}{k+1}\prod_{j=0}^k (n-j)$ of degree $k+1$ at $k+2$ values, so since $\deg P_k(n) \leq k+1$ the two polynomials should agree. Hence we have that

$$P_k(n) = \frac{1}{2(k+1)^2}\binom{2k}{k}\binom{2k+2}{k+1}\prod_{j=0}^{k}(n-j),$$

proving Okada's conjecture (2). We also get that $\deg P_k(n) = k+1$ and so $\deg R_k(n) = k+1$, thereby completing the proof of the lemma. $\square$

We observe now that Okada's conjecture gives us a formula for $\Phi_n(p_k)$, where p_k are the power sum symmetric functions given by $p_k(x_1, \ldots, x_n) = x_1^k + \cdots + x_n^k$. Without loss of generality we can assume that we have only one variable and we will consider $p_k(x) = x^k$. Let $q_k(x) = \prod_{j=1}^k (x - j^2)$ and observe that we can express $p_k(x)$ as a linear combination of $q_k(x), q_{k-1}(x), \ldots, q_0(x)$. Let $p_k = \sum_{i=0}^k A(k,i) q_i$ for some rational coefficients $A(k,i)$. We have that $A(k,i) = 0$ if $i > k$ and $A(0,0) = 1$ assuming $q_0 = 1$, also $A(1,1) = 1$ and comparing coefficients at $[x^k]$ we get that $A(k,k) = 1$ for all k. We now will exhibit a recurrence for the numbers $A(k,i)$ as follows. We have

that

$$p_{k+1} = x^{k+1} = xp_k = x(\sum_{i=0}^{k} A(k,i)q_i) =$$

$$\sum_{i=0}^{k} A(k,i)\left((x-(i+1)^2)+(i+1)^2\right) q_i =$$

$$\sum_{i=0}^{k} A(k,i)q_{i+1} + \sum_{i=0}^{k} A(k,i)(i+1)^2 q_i =$$

$$\sum_{i=1}^{k}(A(k,i-1)+(i+1)^2 A(k,i))q_i + A(k,k)q_{k+1} + A(k,0)q_0.$$

Since the q_is are linearly independent over $\mathbb{Q}$ we get that $A(k+1,k+1) = A(k,k)$ and $A(k+1,i) = A(k,i-1)+(i+1)^2 A(k,i)$. This recurrence is very similar to the one satisfied by the central factorial numbers $T(n,k)$(see exercise 5.8 in [1]), given by $T(n,k) = k^2 T(n-1,k) + T(n-1,k-1)$ and $T(0,0) = 1$, $T(i,j) = 0$ if $i = 0, j > 0$ or $i > 0, j = 0$. In fact we easily see that $A(n,k) = T(n+1,k+1)$ and hence

$$p_k = \sum_{i=0}^{k} T(k+1,i+1)q_i.$$

Thus we obtain the following proposition.

Proposition 1. For $\Phi_n(p_k)$ we have that

$$\Phi_n(p_k) = \frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 \sum_{u\in\lambda} h_u^{2k} = \sum_{i=0}^{k} T(k+1,i+1)\Phi_n(q_i) =$$

$$\sum_{i=0}^{k} T(k+1,i+1)\frac{1}{2(i+1)^2}\binom{2i}{i}\binom{2i+2}{i+1}(i+1)!\binom{n}{i+1}.$$

This result generalizes Han's "marked hook formula" for $\Phi_n(p_1)$, [3, Theorem 1.5].

3 Other similar results

Consider now the case of $F = e_k$, where e_k is the elementary symmetric function given by $e_k(x_1,\ldots,x_n) = \sum_{1\le i_1<i_2<\cdots<i_k\le n} x_{i_1}x_{i_2}\ldots x_{i_n}$. We will show how to find a formula for $\Phi_n(e_k)$. The point is to use the Okounkov-Nekrasov hook length formula [3],

$$\sum_{n\ge 0}\sum_{\lambda\vdash n} x^n \prod_{u\in\lambda}\left(1-\frac{z}{h_u^2}\right) = \prod_{k\ge 1}(1-x^k)^{z-1}. \tag{8}$$

We should point out that the same approach has already been used by Han in [7] to derive the cases for e_1 and e_2 and the following is an extension of his results.

If we make the substitution $1/z = t$ and $y = x/z$ and expand the product over u in the left-hand side of (8) we obtain

$$\sum_{n\geq 0}\sum_{\lambda\vdash n} y^n \prod_{u\in\lambda}\frac{1}{h_u^2}\left(\sum_{j=0}^{n} e_j(\{h_u^2 : u\in\lambda\})t^j(-1)^{n-j}\right) = \prod_{k\geq 1}\left(1-(yt)^k\right)^{1/t-1}. \tag{9}$$

Substituting $\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 e_j(h_u : u\in\lambda)$ with $\Phi_n(e_j)$ we get

$$\sum_{n\geq 0}\frac{y^n}{n!}\sum_j (-1)^{n-j}\Phi_n(e_j)t^j = \prod_{k\geq 1}(1-(yt)^k)^{1/t-1}. \tag{10}$$

So the value of $\Phi_n(e_j)$ is $(-1)^{n-j}n!$ times the coefficient of $y^n t^j$ from the righthand side of (10). We will now expand the right-hand side in a convenient form as follows

$$\begin{aligned}
\prod_{k\geq 1}(1-(yt)^k)^{1/t-1} &= \exp\left((1-1/t)\left(\sum_{k\geq 1} -\log(1-(yt)^k)\right)\right)\\
&= \exp\left((1-1/t)\left(\sum_{k\geq 1, i\geq 1}(yt)^{ki}\right)\right)\\
&= \exp\left((1-1/t)\left(\sum_{m\geq 1}(yt)^m\tau(m)\right)\right)\\
&= \sum_{u\geq 0}\frac{(1-1/t)^u}{u!}\sum_{m_1,\ldots,m_u\geq 1}(yt)^{m_1+\cdots+m_u}\tau(m_1)\cdots\tau(m_u), \qquad (11)
\end{aligned}$$

where $\tau(m)$ is the number of divisors of m. Restricting (11) to the coefficient at y^n is equivalent to imposing the condition $m_1+\cdots+m_u = n$, then restricting further to t^j is equivalent to taking only the term at $\left(\frac{1}{t}\right)^{n-j}$ from $(1-1/t)^u$, so we get that

$$\Phi_n(e_j) = n!(-1)^{n-j}\sum_{u=0}^{n}\frac{1}{u!}\binom{u}{n-j}(-1)^{n-j}\sum_{\substack{m_1+\cdots+m_u=n,\\ m_i\geq 1}}\tau(m_1)\cdots\tau(m_u). \tag{12}$$

Notice that in order for $\binom{u}{n-j}\neq 0$ we would need $u\geq n-j$, so we can write $q = n-u$, going from 0 to j, and then further substitute $m_i = a_i+1$, $a_i\geq 0$, so that $\sum_{i=1}^{n-q} a_i = q$. Thereby we get that

$$\Phi_n(e_j) = \sum_{q=0}^{j}\frac{n!}{(n-j)!(j-q)!}\sum_{\substack{a_1+\cdots+a_{n-q}=q,\\ a_i\geq 0}}\tau(a_1+1)\cdots\tau(a_{n-q}+1).$$

Notice that the unordered solutions $(a_1,\ldots,a_{n-q})$ of $a_1+\cdots+a_{n-q} = q$ are in bijection with the choice of $p\leq q$ of the a_is to be nonzero. If we label those nonzero a_is by b_ks for $k = 1,\ldots,p$ we obtain the following

Proposition 2.

$$\Phi_n(e_j) = \binom{n}{j} \sum_{q=0}^{j} \frac{j!}{(j-q)!} \sum_{p=0}^{q} \binom{n-q}{p} \sum_{b_1+\cdots+b_p=q, b_i\geq 1} \tau(b_1+1)\cdots\tau(b_p+1).$$

We also observe that for a fixed j $\Phi_n(e_j)$ is indeed a polynomial in n.

We will now exhibit a more general upper bound for the degree of $\Phi_n(p_\mu)$, where $\mu = (\mu_1, \ldots, \mu_j) \vdash k$ with $\mu_j \neq 0$ and we use the power sum symmetric function p_μ as F. In this case we have that

$$\begin{aligned}
\Phi_n(p_\mu) &= \frac{1}{n!} \sum_{\lambda\vdash n} f_\lambda^2 \left(\sum_{u\in\lambda} h_u^{2\mu_1}\right) \ldots \left(\sum_{u\in\lambda} h_u^{2\mu_j}\right) \\
&\leq \frac{1}{n!} \sum_{\lambda\vdash n} f_\lambda^2 (n \max(h_u : u\in\lambda)^{2\mu_1}) \cdots (n \max(h_u : u\in\lambda)^{2\mu_j}) \\
&= n^j \frac{1}{n!} \sum_{\lambda\vdash n} f_\lambda^2 \max(h_u : u\in\lambda)^{2k}.
\end{aligned}$$

By the proof of Lemma 2 we have that $\frac{1}{n!}\sum_{\lambda\vdash n} f_\lambda^2 \max(h_u : u\in\lambda)^{2k} \sim n^k$, so we get that

$$\deg \Phi_n(p_\mu) \leq j + k.$$

References

[1] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999. MR **1676282 (2000k:**05026)

[2] ______, *Some Combinatorial properties of hook lengths, contents, and parts of partitions*, arXiv:0807.0383v1.

[3] Guo-Niu Han, *The Nekrasov-Okounkov hook length formula: refinement, elementary proof, extension and applications*, arXiv: 0805.1398v1.

[4] Richard P. Stanley, *Increasing and decreasing subsequences and their variants*, International Congress of Mathematicians. Vol. I, Eur. Math. Soc., Zürich, 2007, pp. 545–579. MR **2334203 (2008d:**05002)

[5] J. M. Hammersley, *A few seedlings of research*, Proc. Sixth Berkeley Symp. on Math. Statist. and Prob., Vol. 1 (Univ. California, Berkeley, Calif., 1970/1971), Univ. California Press, Berkeley, Calif., 1972, pp. 345–394. MR 0405665 (53 #9457)

[6] Guo-Niu Han, *Some conjectures and open problems on partition hook lengths*, to appear in Experimental Mathematics.

[7] ______, *An explicit expansion formula for the powers of the Euler Product in terms of partition hook lengths*, arXiv:0804.1849v3.